Fourth-Order Finite Difference Analogues of the Dirichlet Problem for Poisson's Equation in Three and Four Dimensions

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I. Introdution. In a recent paper [1] Bramble and Hubbard formulated finite difference analogues of the Dirichlet problem for Poisson's equation in the plane which were $O(h^4)$, h being the mesh width. Subsequently in [2] they gave a general theorem on error estimation for a class of finite difference analogues to the Dirichlet problem for some general uniformly elliptic equations in N-dimensions. Some examples in the plane are formulated there.

In dealing with Poisson's equation by finite difference methods a very large system of linear equations must be solved. Even with modern high-speed computers this number may be prohibitively large if the desired accuracy is to be obtained. Several methods commonly used in plane problems are $O(h^2)$ and their direct analogues in higher dimensions can also be shown to be $O(h^2)$. Thus, if (for smooth problems) in three dimensions a fourth-order method were used instead of a second-order one, it might be expected that a considerably smaller system would yield comparable accuracy. Consequently, if a higher order method were used some problems might move to within the range of practical feasibility.

In this paper analogues to the Dirichlet problem for Poisson's equation in three and four dimensions are given. These analogues are shown to be $O(h^4)$ as $h \to 0$.

2. Three Dimensional Case. Let R be a bounded region with boundary C in three dimensions. In the usual manner the space is subdivided into cubes of side h with faces parallel to the (x, y), (x, z), and (y, z) planes. The corner points of these cubes will be called mesh points. The set R_h will consist of those mesh points P in R whose 18 nearest neighboring mesh points, and the lines joining them to P, are in R. The set C_h^{**} will denote those mesh points $P \in R - R_h$ whose 6 nearest neighbors and the lines joining them to P are in R. The set of mesh points in $R - R_h - C_h^{**}$ will be called C_h^*. If P is in C_h^* then at least one line joining P to one of its 6 nearest neighbors say $(x - h, y, z)$ is cut by C. Thus for some α, $0 < \alpha \leqq 1$, $(x - \alpha h, y, z)$ is on C. Such a point will be said to lie in C_h. Similarly, one of the neighbors of (x, y, z) in the y and z directions may not be in R. These points will also then lie in C_h. The totality of such "neighbors" of points of C_h^* will make up the set C_h. The mesh size is assumed so small that if (x, y, z) is in C_h^* and $(x \pm \alpha h, y, z)$ is in C_h then $(x \pm h, y, z)$ and $(x \pm 2h, y, z)$ are in $R + C$ where either the plus sign is taken at each of the points or the minus sign is taken. Analogous statements are assumed for the y and z directions.

With the preceding sets defined we are in a position to formulate the finite

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difference problem. The exact problem to be approximated is

$$\begin{aligned} \Delta u &= F \quad \text{in} \quad \mathrm{R} \\ u &= f \quad \text{on} \quad C, \end{aligned} \tag{2.1}$$

where Δ is the Laplace operator, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, and F and f are sufficiently smooth functions defined in R and on C respectively.

At a point (x, y, z) of R_h we approximate Δu by

$$\begin{aligned} \Delta_h u(x, y, z) = \frac{1}{6h^2} \{&2[u(x + h, y, z) + u(x - h, y, z) + u(x, y + h, z) \\ &+ u(x, y - h, z) + u(x, y, z + h) + u(x, y, z - h)] + u(x + h, y + h, z) \\ &+ u(x + h, y - h, z) + u(x - h, y + h, z) + u(x - h, y - h, z) \\ &+ u(x + h, y, z + h) + u(x + h, y, z - h) + u(x - h, y, z + h) \\ &+ u(x - h, y, z - h) + u(x, y + h, z + h) + u(x, y + h, z - h) \\ &+ u(x, y - h, z + h) + u(x, y - h, z - h) - 24u(x, y, z)\}. \end{aligned} \tag{2.2}$$

By approximating $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ by means of

$$\begin{aligned} \Delta^{+}_{h(x,y)} u \equiv \frac{1}{h^2} [&u(x + h, y, z) + u(x - h, y, z) \\ &+ u(x, y + h, z) + u(x, y - h, z) - 4u(x, y)] \end{aligned}$$

and

$$\begin{aligned} \Delta^{\times}_{h(x,y)} u \equiv \frac{1}{2h^2} [&u(x + h, y + h, z) + u(x + h, y - h, z) \\ &+ u(x - h, y + h, z) + u(x - h, y - h, z) - 4u(x, y)] \end{aligned}$$

with similar considerations in the (x, z) and (y, z) planes it is easy to see that $\Delta_h u$ given by (2.2) is just

$$\Delta_h u = \tfrac{1}{2}\{ \Delta_{h(x,y)} u + \Delta_{h(x,z)} u + \Delta_{h(y,z)} u\},$$

where

$$\Delta_{h(x,y)} u = \tfrac{1}{3}\Delta^{+}_{h(x,y)} u + \tfrac{2}{3}\Delta^{\times}_{h(x,y)} u.$$

From this structure it is not difficult to see that

$$\left| \Delta_h u - \left(\Delta u + \frac{h^2}{6} \Delta^2 u\right) \right| \leqq \frac{M_6}{10} h^4, \tag{2.3}$$

where M_i is a uniform bound for any ith partial derivative of u in $R + C$.

At a point of C_h^{**} we define

$$\begin{aligned} \Delta_h^{**} u = \frac{1}{h^2} [&u(x + h, y, z) + u(x - h, y, z) + u(x, y + h, z) \\ &+ u(x, y - h, z) + u(x, y, z + h) + u(x, y, z - h) - 6u(x, y, z)]. \end{aligned} \tag{2.4}$$

The inequality

$$(2.5) \qquad | \Delta_h^{**}u - \Delta u | \leqq \frac{M_4}{4} h^2$$

holds at points of C_h^{**}.

At a point of C_h^* the pure second partial derivatives are approximated to $O(h^2)$, if necessary by an unbalanced four-point formula which includes a point of C_h. (See [1]). For example, if (x, y, z) is in C_h^* and $(x - \alpha h, y, z) \in C_h$ then we have that

$$(2.6) \qquad \left| u_{xx}(x, y, z) - h^{-2}\left\{\left(\frac{\alpha - 1}{\alpha + 2}\right) u(x + 2h, y, z) + \frac{2(2 - \alpha)}{\alpha + 1} u(x + h, y, z) + \frac{6}{\alpha(\alpha + 1)(\alpha + 2)} u(x - \alpha h, y, z) - \frac{3 - \alpha}{\alpha} u(x, y, z)\right\}\right| \leqq \frac{M_4}{6} h^2 + \frac{M_5}{6} h^3.$$

If the neighbors of (x, y, z) in the y and z direction are in R then we define on C_h^*

$$(2.7) \qquad \begin{aligned} \Delta_h^* u = \frac{1}{h^2}\Big\{&\frac{\alpha - 1}{\alpha + 2} u(x + 2h, y, z) + \frac{2(2 - \alpha)}{\alpha + 1} u(x + h, y, z) \\ &+ \frac{6}{\alpha(\alpha + 1)(\alpha + 2)} u(x - \alpha h, y, z) + u(x, y + h, z) + u(x, y - h, z) \\ &+ u(x, y, z + h) + u(x, y, z - h) - 3\,\frac{\alpha + 1}{\alpha} u(x, y, z)\Big\}. \end{aligned}$$

It is easy to see that

$$(2.8) \qquad | \Delta_h^* u - \Delta u | \leqq \frac{M_4}{2} h^2 + \frac{M_5}{2} h^3.$$

(See [1]). At each point of C_h^*,Δ_h^* is defined analogously, using the four-point approximation when needed.

As our approximating problem we consider the following linear system

$$(2.9) \qquad \begin{aligned} \Delta_h U(P) &= F(P) + \frac{h^2}{6} \Delta F(P), && P \in R_h \\ \Delta_h^{**}U(P) &= F(P), && P \in C_h^{**} \\ \Delta_h^*U(P) &= F(P), && P \in C_h^* \\ U(P) &= f(P), && P \in C_h \end{aligned}$$

for the determination of U at the points of $R_h + C_h^* + C_h^{**}$. The system (2.9) is not of positive type (see e.g., Forsythe and Wasow [3]) however, it does have the properties of "interior positivity" and "strict diagonal dominance". (See [1]). As in the case of the plane [1] these conditions will suffice to show that if

$$\epsilon(P) = u(P) - U(P)$$

then,

$$(2.10) \qquad | \epsilon(P) |_M = O(h^4),$$

where the subscript M denotes the maximum over all $P \in R_h + C_h^* + C_h^{**}$.

The method of proof follows closely that given by Bramble and Hubbard in [1].

Let $G_h(P, Q)$ be the "Green's function" defined by

$$\begin{aligned} \Delta_{h,P}G_h(P, Q) &= -h^{-3}\delta(P, Q), \qquad P \in R_h \\ \Delta_{h,P}^{**}G_h(P, Q) &= -h^{-3}\delta(P, Q), \qquad P \in C_h^{**} \\ G_h(P, Q) &= \delta(P, Q), \qquad P \in C_h^{*}, \end{aligned} \tag{2.11}$$

for each $Q \in R_h + C_h^{**} + C_h^{*}$. By the usual means it may be shown that $G_h(P, Q) \geqq 0$. It may be easily verified that any mesh function $V(P)$ defined for $P \in R_h + C_h^{**} + C_h^{*}$ satisfies the identity

$$\begin{aligned} V(P) = h^3 \sum_{Q \in R_h} G_h(P, Q)[-\Delta_h V(Q)] \\ + h^3 \sum_{Q \in C_h^{**}} G_h(P, Q)[-\Delta_h^{**} V(Q)] + \sum_{Q \in C_h^{*}} G_h(P, Q) V(Q). \end{aligned} \tag{2.12}$$

In particular, if we take

$$V(P) = 1, \qquad P \in R_h + C_h^{**}$$

and

$$V(P) = 0 \quad \text{for} \quad P \in C_h^{*}$$

then we obtain

$$1 \geqq h \sum_{Q \in C_h^{**}} G_h(P, Q). \tag{2.13}$$

Because of the interior positivity of (2.9) it can be seen that if

$$\begin{aligned} \Delta_h W(P) \geqq 0, \qquad P \in R_h \\ \Delta_h^{**} W(P) \geqq 0, \qquad P \in C_h^{**} \end{aligned} \tag{2.14}$$

then

$$W(Q) \leqq \max_{P \in C_h^{*}} W(P), \qquad Q \in R_h + C_h^{**} + C_h^{*}.$$

This is just an interior maximum principle. By making use of (2.14) it can be readily shown as was done in [1] that

$$h^3 \sum_{Q \in R_h} G_h(P, Q) \leqq \frac{d^2}{24}, \tag{2.15}$$

where d is the diameter of R.

Let us now apply (2.12) to $\epsilon(P)$. Making use of the (2.13) and (2.15) and the fact that $G_h(P, Q) \geqq 0$ we have that

$$\begin{aligned} |\epsilon(P)| \leqq \frac{d^2}{24} [\max_{Q \in R_h} |\Delta_h \epsilon(Q)|] \\ + h^2[\max_{Q \in C_h^{**}} |\Delta_h^{**}\epsilon(Q)|] + \sum_{Q \in C_h^{*}} G_h(P, Q) |\epsilon(Q)|. \end{aligned} \tag{2.16}$$

From (2.3), (2.5) and (2.9) it follows that

$$|\epsilon(P)| \leqq \left[\frac{d^2}{240} M_6 + \frac{M_4}{4}\right] h^4 + \sum_{Q \in C_h^{*}} G_h(P, Q) |\epsilon(Q)|. \tag{2.17}$$

Now if we use the definition of Δ_h^* ((2.7) or the appropriate analogue of (2.7)) and the fact that $\epsilon(P) = 0, \quad P \in C_h$ we find that

$$(2.18) \qquad |\epsilon(P)| \leqq \frac{5}{6}|\epsilon|_M + \frac{h^2}{6}|\Delta_h^*\epsilon(P)|, \qquad P \in C_h^*.$$

But, from (2.9) and (2.8)

$$(2.19) \qquad |\Delta_h^*\epsilon(P)| = |\Delta_h^*u(P) - \Delta u(P)| \leqq \frac{M_4}{2}h^2 + \frac{M_5}{2}h^3.$$

Hence, combining (2.18) and (2.19), we have

$$(2.20) \qquad |\epsilon(P)| \leqq \frac{5}{6}|\epsilon|_M + \left(\frac{M_4}{12} + \frac{M_5}{12}h\right)h^4, \qquad P \in C_h^*$$

Now, since $\sum_{Q \in C_h^*} G_h(P, Q) \leqq 1$, we have from (2.17) and (2.20)

$$(2.21) \qquad |\epsilon(P)| \leqq \left[\frac{d^2}{240}M_6 + \frac{h}{12}M_5 + \frac{1}{3}M_4\right]h^4 + \frac{5}{6}|\epsilon|_M.$$

Since the right hand side of (2.21) is independent of P we conclude that

$$(2.22) \qquad |\epsilon|_M \leqq \left[\frac{d^2}{40}M_6 + \frac{h}{2}M_5 + 2M_4\right]h^4.$$

This shows that the overall error produced in replacing problem (2.1) by (2.9) is $O(h^4)$.

3. Higher Dimensional Problems. Let us assume that the sets R_h, C_h^{**}, C_h^*, and C_h have been defined in a manner analogous to that of the preceding section.

In formulating $O(h^4)$ analogues to (2.1) in N dimensions we could use the direct analogues of (2.4) and (2.7). The problem reduces to that of finding the analogue of (2.2) at a point $(x_1, \cdots, x_N) \in R_h$.

Let us proceed as described after (2.2) and consider various two dimensional planes through P and the Laplace difference operators in these respective planes. It turns out that if we define

$$(3.1) \qquad \Delta_h u = \frac{1}{N-1}\left\{\sum_{\substack{i=1,\cdots,N-1\\ j=2,\cdots,N\\ j>i}}\left[\frac{4-N}{3}\Delta^{+}_{h(x_i,x_j)}u + \frac{N-1}{3}\Delta^{\times}_{h(x_i,x_j)}u\right]\right\}$$

then the relation

$$(3.2) \qquad \left|\Delta_h u - \left(\Delta u + \frac{N-1}{6}h^2\Delta^2 u\right)\right| = O(h^4)$$

is valid in R_h. In order that (3.1) be of positive type (see, e.g., [3]) it is necessary that $N \leqq 4$. For $N = 2$, (3.1) is the usual nine-point formula in the plane and for $N = 3$, (3.1) reduces to (2.2) of Section II. The case $N = 4$ is interesting in that

$$(3.3) \qquad \Delta_h u = \frac{1}{3}\left\{\sum_{\substack{i=1,\cdots,3\\ j=2,\cdots,4\\ j>i}}\Delta^{\times}_{h(x_i,x_j)}u\right\},$$

and the term involving $\Delta^{+}_{h(x_i,x_j)}$ drops out. Thus, in three dimensions (2.2) is a 19-point formula while in four dimensions (3.3) is a 25-point formula, both being $O(h^4)$ expressions.

For $N > 4$, although (3.3) is $O(h^4)$ locally, it is not of positive type. Thus, the method of Section III is not applicable and it is not clear that an overall $O(h^4)$ estimate for the truncation error $\epsilon(P)$ would result. It seems that a different approach might be more desirable for $N > 4$.

It should be noted that if, at points of C_h^*, the direct analogue of the Shortley and Weller approximation [4] is used, then an overall $O(h^3)$ estimate for the truncation error could be obtained in two, three or four dimensions.

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1. J. H. BRAMBLE, & B. E. HUBBARD, "On the formulation of finite difference analogues of the Dirichlet problem for Poisson's equation," *Numer. Math.*, v. 4, 1962, p. 313–327.

2. J. H. BRAMBLE, & B. E. HUBBARD, "A theorem on error estimation for finite difference analogues of the Dirichlet problem for elliptic equations," (to appear).

3. G. FORSYTHE, & W. WASOW, *Finite Difference Methods for Partial Difference Equations*, New York, Wiley, 1960.

4. G. SHORTLEY & R. WELLER, "The numerical solution of Laplace's equation," *J. Appl. Phys.*, v. 9, 1938, p. 334–348.